MATH 3236 Statistical Theory

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Chi square distribution.

$\chi^2_m \sim \Gamma\left(\frac{m}{2}, \frac{1}{2}\right)$

If X_i are independent $N(0,1)$

$$\sum_{i=1}^{m} X_i^2 \sim \chi^2_m$$

If you have a sample X_i from a population that is $N(\mu, \sigma^2)$

$$\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2_n$$

μ is Known

$$\hat{\sigma}^2 = \frac{1}{N} \sum_{i=1}^{N} (X_i - \mu)^2$$

$$\frac{N\hat{\sigma}^2}{\sigma^2} = \sum_{i=1}^{N}\left(\frac{X_i - \mu}{\sigma}\right)^2 \simeq \chi^2_N$$

$$\mathbb{P}\left(\sigma^2 < 1.1\,\hat{\sigma}^2\right) =$$

$$\mathbb{P}\Big(\underbrace{\frac{N\hat{\sigma}^2}{\sigma^2}}_{\chi^2_N} > N \cdot 1.1\Big)$$

if X is χ^2_N $\quad \mathbb{E}(X) = N$

X_i is a sample from a $\mathcal{N}(\mu, \sigma^2)$ $\quad \mu \quad \sigma^2$ Known

$$\frac{1}{N}\sum_i (X_i - \mu)^2$$

$$\mathbb{P}\left(\frac{1}{N}\sum_i (X_i - \mu)^2 \geq u^2\right)$$

$$= \mathbb{P}\left(\frac{\sum_i (X_i - \mu)^2}{\sigma^2} \geq \frac{N\mu^2}{\sigma^2} \right)$$

o

X_i are exponential λ

α, β

$\alpha + N$ $\quad$ $\beta + \sum_i X_i$

$$\hat{\lambda}(\underline{X}) = \frac{\alpha + N}{\beta + \sum_i X_i}$$

$$\mathbb{P}(|\hat{\lambda} - \lambda| < 0.1) =$$

$$\int \mathbb{P}(|\hat{\lambda} - \lambda| < 0.1 \mid \lambda)\, \mathbb{P}(\lambda)\, d\lambda$$

posterior

$$= \mathbb{E}_\lambda\left(\mathbb{P}(|\hat{\lambda} - \lambda| < 0.1 \mid \lambda) \right)$$

$\hat{\lambda}$ c.d.f. F

$$\hat{\lambda} = \frac{\alpha + N}{\beta + \sum_{i=1}^{N} T_i} \qquad T_i \ \exp \ \lambda$$

$$\sum_{i=1}^{N} T_i = \Gamma(N, \lambda)$$

$$F(t \mid \lambda) = \mathbb{P}(\hat{\lambda} \leq t) = \mathbb{P}\left(\frac{\alpha + N}{\beta + \sum_i T_i} \leq t\right) =$$

$$= \mathbb{P}\left(\sum_i T_i \geq \frac{\alpha + N}{t} - \beta\right) =$$

$$= 1 - G\left(\frac{\alpha + N}{t} - \beta\right)$$

where G is the c.d.f. of $\Gamma(N, \lambda)$

$$\mathbb{P}(|\hat{\lambda} - \lambda| < 0.1 \mid \lambda)$$

$$F(\lambda + 0.1 \mid \lambda) - F(\lambda - 0.1 \mid \lambda)$$

Bayes point of view is quite

complex to implement.

Maximum Likelihood.

$$\hat{\lambda} = \frac{N}{\sum X_i} = \frac{N}{T}$$

$$\mathbb{P}\left(|\hat{\lambda} - \lambda| > 0.1 \right) = ?$$

$$\mathbb{P}\left(|\hat{\lambda} - \lambda| > 0.1\lambda \right)$$

T is $\Gamma(N, \lambda) \implies$

λT is $\Gamma(N, 1)$

$$\mathbb{P}\left(\left| \frac{\hat{\lambda}}{\lambda} - 1 \right| \geq 0.1 \right) =$$

$$\mathbb{P}\left(\left| \frac{N}{\lambda T} - 1 \right| \geq 0.1 \right)$$

$\lambda T = Z$ The distribution of Z does depend on λ.

$$\mathbb{P}\left(\left|\frac{N}{Z}-1\right| \geq 0.1\right)$$

$$1-\mathbb{P}\left(\frac{N}{1.1} \leq Z \leq \frac{N}{0.9}\right)=1-G\left(\frac{N}{0.9}\right)+G\left(\frac{N}{1.1}\right)$$

G is the c.d.f. of exp $N, 1$.

X_i are $N(\mu, \sigma^2)$

$$\sum_i \left(\frac{X_i-\mu}{\sigma}\right)^2 \simeq \chi^2_n$$

X_i are exp λ

$$\lambda \sum_{i=1}^{N} X_i \simeq \Gamma(N, 1)$$

pivotal quantity.

X_i normal μ, σ^2

σ^2 is known μ is unknown

$$\hat{\mu} = \frac{1}{N} \sum_{i=1}^{N} X_i$$

$$\boxed{\frac{1}{\sqrt{N}} \sum_i \frac{X_i - \mu}{\sigma} \simeq N(0,1)}$$ pivotal quantity

$$\mathbb{P}\left(|\hat{\mu} - \mu| > 0.1 \right) =$$

$$\hat{\mu} - \mu = \frac{1}{N} \sum_i (X_i - \mu)$$

$$= \mathbb{P}\left(\left| \frac{\hat{\mu} - \mu}{\sigma/\sqrt{N}} \right| > \frac{0.1}{\sigma/\sqrt{N}} \right) =$$

$$= \mathbb{P}\left(-\frac{0.1}{\sigma/\sqrt{N}} < Z < 0.1/\sigma/\sqrt{N} \right) =$$

$$Z = N(0,1)$$

$$= \Phi\left(\frac{0.1\sqrt{N}}{\sigma}\right) - \Phi\left(-\frac{0.1\sqrt{N}}{\sigma}\right)$$

X_i are $\mathcal{N}(\mu, \sigma^2)$

$$\bar{X} = \frac{1}{N} \sum_{i=1}^{N} X_i = \hat{\mu}$$

$$\hat{\sigma}^2 = \frac{1}{N} \sum_i (X_i - \bar{X})^2$$

$$\mathbb{E}(\hat{\sigma}^2) = \frac{N-1}{N} \sigma^2$$

$$\hat{\sigma}^2 = \frac{1}{N-1} \sum_i (X_i - \bar{X})^2$$ unbiased version

Theorem: $X_1 \dots X_N$ are i.i.d. $\mathcal{N}(0, 1)$ Then

$$\bar{X} = \frac{1}{N} \sum_i X_i \qquad \hat{\sigma}^2 = \frac{1}{N} \sum_{i=1}^{N} (X_i - \bar{X})^2$$

are independent and

$\bar{X}$ is $N\left(0, \frac{1}{\sqrt{N}}\right)$

$N \hat{\sigma}^2$ is χ^2_{N-1}

X_i $N(\mu, \sigma^2)$

$$\bar{X} \simeq N\left(\mu, \frac{\sigma^2}{N}\right)$$

$$N \frac{\hat{\sigma}^2}{\sigma^2} \simeq \chi^2_{N-1}$$

$X_i - \bar{X}$ contains only $N-1$ indep quantities

$$\sum_i (X_i - \bar{X}) = 0$$
